1. Consider the left network below, where an edge annotation 3/4 means capacity = 4, flow = 3. The source is vertex s; the sink is vertex t. On the vertex pattern to the right, draw the residual graph $G_f$ that the Ford-Fulkerson algorithm associates with the given flow $f$ on the graph $G$ to the left.

![Network Diagram](image)

2. Specify a vertex sequence that constitutes an augmenting path from $s$ to $t$ in the residual graph.

3. Let $P(n)$ be the number of ways to parenthesize a chain of $n$ matrices for multiplication. We have $P(1) = P(2) = 1$ and $P(3) = 2$. For $n > 3$, write a recurrence for $P(n)$ as a expression involving $P(k)$, for $1 \leq k < n$.

4. Suppose matrices $A_1, A_2, \ldots, A_n$ has respective dimensions $p_0 \times p_1, p_1 \times p_2, \ldots, p_{n-1} \times p_n$ and are therefore compatible for multiplication in the given order. For $i \leq j$, let $m[i,j]$ be the minimum number of multiplications necessary to compute the product $A_i A_{i+1} \cdots A_j$. Write a recurrence that expresses $m[i,j]$ as a function of $m[i,k]$ and $m[k+1,j]$ for $i \leq k < j$. 

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Solutions.

1. Consider the left network below, where an edge annotation $3/4$ means capacity $= 4$, flow $= 3$. The source is vertex $s$; the sink is vertex $t$. On the vertex pattern to the right, draw the residual graph $G_f$ that the Ford-Fulkerson algorithm associates with the given flow $f$ on the graph $G$ to the left.

2. Specify a vertex sequence that constitutes an augmenting path from $s$ to $t$ in the residual graph. Augmenting paths include
   - $s \to b \to a \to t$
   - $s \to f \to e \to t$
   - $s \to f \to t$

3. Let $P(n)$ be the number of ways to parenthesize a chain of $n$ matrices for multiplication. We have $P(1) = P(2) = 1$ and $P(3) = 2$. For $n > 3$, write a recurrence for $P(n)$ as a expression involving $P(k)$, for $1 \leq k < n$.

   $$P(n) = \sum_{k=1}^{n-1} P(k)P(n-k).$$

4. Suppose matrices $A_1, A_2, \ldots, A_n$ has respective dimensions $p_0 \times p_1, p_1 \times p_2, \ldots, p_{n-1} \times p_n$ and are therefore compatible for multiplication in the given order. For $i \leq j$, let $m[i, j]$ be the minimum number of multiplications necessary to compute the product $A_i A_{i+1} \cdots A_j$. Write a recurrence that expresses $m[i, j]$ as a function of $m[i, k]$ and $m[k+1, j]$ for $i \leq k < j$.

   $$m[i, j] = \begin{cases} 0, & i = j \\ \min_{k \leq j} \{m[i, k] + m[k+1, j] + p_{i-1}p_kp_j\}, & i < j \end{cases}$$