Circle T (true) or F (false) as appropriate. Each of the 17 true/false questions is worth 1 point. Question 4 is worth 3 points.

1. Suppose \( v \) is a descendant of \( u \) in a \( G_\pi \)-tree created by the depth-first-search algorithm. How are the discovery and finish times of \( u \) and \( v \) related? Recall \( w.d \) denotes the discovery time of node \( w \), while \( w.f \) denotes its finish time.

(a) F T \( u.d < v.d < v.f < u.f \)  
(b) F T \( v.d < u.d < u.f < v.f \)  
(c) F T \( v.d < v.f < u.d < u.f \)  
(d) F T \( u.d < u.f < v.d < v.f \)  
(e) F T \( v.d < u.d < v.f < u.f \)

2. Consider the point in the depth-first search algorithm when vertex \( v \) is observed on the adjacency list of vertex \( u \). Then \( (u, v) \) is a tree edge if \( v \) is

(a) F T white  
(b) F T gray  
(c) F T black  
(d) F T colorless

3. Consider the point in the depth-first search algorithm when a gray vertex \( v \) is observed on the adjacency list of vertex \( u \). Then

(a) F T \( (u, v) \) is a forward edge  
(b) F T \( (u, v) \) is a backward edge  
(c) F T \( (u, v) \) is a cross edge  
(d) F T \( (u, v) \) is a tree edge

4. Suppose a directed acyclic graph contains the edge \( (v, u) \). Why, when we run depth-first-search on the graph, must we obtain \( u.f < v.f \)?

5. Suppose \( G_n = (V_n, E_n) \) is a sequence of graphs of increasing size. That is, \(|V_{n+1}| + |E_{n+1}| > |V_n| + |E_n|\). Let \( f(n) \) be the size of the adjacency list representation for \( G_n \). Let \( g(n) \) be the size of the adjacency matrix representation for \( G_n \).

(a) F T If \( |E_n| = \Theta(|V_n|) \), then \( f = \Theta(g) \).  
(b) F T If \( |E_n| = \Theta(|V_n|^2) \), then \( f = \Theta(g) \).  
(c) F T If \( |E_n| = \Theta(\sqrt{|V_n|}) \), then \( f = O(g) \).  
(d) F T If \( |E_n| = \Theta(|V_n|^{3/2}) \), then \( f = \omega(g) \).
Solutions.

1. Suppose $v$ is a descendant of $u$ in a $G_\pi$-tree created by the depth-first-search algorithm. How are the discovery and finish times of $u$ and $v$ related? Recall $w.d$ denotes the discovery time of node $w$, while $w.f$ denotes its finish time.

   (a) **True** $u.d < v.d < v.f < u.f$
   (b) **False** $v.d < u.d < u.f < v.f$
   (c) **False** $v.d < v.f < u.d < u.f$
   (d) **False** $u.d < u.f < v.d < v.f$
   (e) **False** $v.d < u.d < v.f < u.f$

   From the parentheses theorem, (a) is **true** and (b) - (e) are **false**.

2. Consider the point in the depth-first search algorithm when vertex $v$ is observed on the adjacency list of vertex $u$. Then $(u, v)$ is a tree edge if $v$ is

   (a) **True** white
   (b) **False** gray
   (c) **False** black
   (d) **False** colorless

   From the dynamic edge classification theorem, $(u, v)$ is a tree edge when $v$ is white. (a) is **true**; (b) - (d) are **false**.

3. Consider the point in the depth-first search algorithm when a gray vertex $v$ is observed on the adjacency list of vertex $u$. Then

   (a) **False** $(u, v)$ is a forward edge
   (b) **True** $(u, v)$ is a backward edge
   (c) **False** $(u, v)$ is a cross edge
   (d) **False** $(u, v)$ is a tree edge

   From the dynamic edge classification theorem, $(u, v)$ is a backward edge when $v$ is gray. (b) is **true**; (a), (c), and (d) are **false**.

4. Suppose a directed acyclic graph contains the edge $(v, u)$. Why, when we run depth-first-search on the graph, must we obtain $u.f < v.f$?

   Assume, for purposes of deriving a contradiction, that we obtain $v.f < u.f$. There are three possible locations for $u.d$.

   (a) We find $v.d < v.f < u.d < u.f$. This case is not possible because $u$ is on the adjacency list of $v$. Hence, $u$ will be seen before $v.f$, and if $u$ has not already been discovered, it will be discovered at that point — before $v.f$.
   (b) We find $v.d < u.d < v.f < u.f$. This case is not possible because the parenthesis theorem precludes interleaving of discovery-finish intervals.
   (c) We find $u.d < v.d < v.f < u.f$. In this case, the parenthesis theorem assures us that $v$ is a $G_\pi$ descendant of $u$, which means that the edge $(v, u)$ is a back edge. However, an acyclic graph can have no back edges because such an edge would induce a cycle.

   We conclude that DFS must find $u.f < v.f$.
5. Suppose \( G_n = (V_n, E_n) \) is a sequence of graphs of increasing size. That is, \(|V_{n+1}| + |E_{n+1}| > |V_n| + |E_n|\). Let \( f(n) \) be the size of the adjacency list representation for \( G_n \). Let \( g(n) \) be the size of the adjacency matrix representation for \( G_n \).

(a) **False** If \(|E_n| = \Theta(|V_n|)\), then \( f = \Theta(g) \).

(b) **True** If \(|E_n| = \Theta(|V_n|^2)\), then \( f = \Theta(g) \).

(c) **True** If \(|E_n| = \Theta(\sqrt{|V_n|})\), then \( f = O(g) \).

(d) **False** If \(|E_n| = \Theta(|V_n|^{3/2})\), then \( f = \omega(g) \).

We have \( K_1(|V_n| + |E_n|) \leq f(n) \leq K_2(|V_n| + |E_n|) \) and \( K_3|V_n|^2 \leq g(n) \leq K_4|V_n|^2 \), which implies
\[
\frac{K_1(|V_n| + |E_n|)}{K_4|V_n|^2} \leq \frac{f(n)}{g(n)} \leq \frac{K_2(|V_n| + |E_n|)}{K_3|V_n|^2}.
\]

(a) For this case, we have \(|E_n| \leq K_5|V_n|\), which then implies
\[
\frac{f(n)}{g(n)} \leq \frac{K_2(|V_n| + K_5|V_n|)}{K_3|V_n|^2} = \frac{K_2 + K_2K_5}{K_3|V_n|} \leq 0.
\]
So \( f \neq \Omega(f) \), and therefore \( f \neq \Theta(g) \). This case is **false**.

(b) For this case, we have \( K_6|V_n|^2 \leq |E_n| \leq K_7|V_n|^2 \), which implies
\[
\frac{K_1(|V_n| + K_6|V_n|^2)}{K_4|V_n|^2} \leq \frac{f(n)}{g(n)} \leq \frac{K_2(|V_n| + K_7|V_n|^2)}{K_3|V_n|^2} \geq \frac{K_2 + K_2K_7}{K_3}.
\]
Therefore, \( f = \Theta(g) \). This case is **true**.

(c) For this case, we have \(|E_n| \leq K_8|V_n|^{1/2} \), which implies
\[
\frac{f(n)}{g(n)} \leq \frac{K_2(|V_n| + K_8|V_n|^{1/2})}{K_3|V_n|^2} \leq \frac{(K_2 + K_2K_8)|V_n|}{K_3|V_n|^2} = \frac{K_2 + K_2K_8}{K_3|V_n|} \leq 0.
\]
So \( f = o(g) \), which implies \( f = O(g) \). This case is **true**.

(d) For this case, we have \(|E_n| \leq K_9|V_n|^{3/2} \), which implies
\[
\frac{f(n)}{g(n)} \leq \frac{K_2(|V_n| + K_9|V_n|^{3/2})}{K_3|V_n|^2} \leq \frac{(K_2 + K_2K_9)|V|^{3/2}}{K_3|V_n|^2} = \frac{K_2 + K_2K_9}{K_3|V_n|^{1/2}} \leq 0.
\]
Therefore \( f = o(g) \). Then \( f \neq \omega(g) \). This case is **false**.