Solutions

1. For a fixed integer \( k > 0 \) and a fixed real \( \epsilon > 0 \), find
\[
\lim_{n \to \infty} \frac{(\ln n)^k}{n^\epsilon}.
\]

Solutions. Via L'Hôpital,
\[
\frac{(\ln n)^k}{n^\epsilon} \to \frac{k!(\ln n)^{k-1}}{\epsilon n^{\epsilon-1}} \cdot \frac{1}{n^\epsilon} \to 0.
\]

Find the asymptotic complexity class of \( T(n) \) for each of the following recurrences. That is, complete the expression \( T(n) = \Theta(?) \). In each case, prove your claim using the master template or other recurrence techniques in case the master template is not applicable. The notation \( \lg n \) refers to \( \log_2 n \). Assume that expressions that should evaluate to integers contain an implied round-down function. For example, \( T(n/3) \) means \( T(\lfloor n/3 \rfloor) \) when necessary.

2. \( T(n) = 2T(n/4) + n^2 \)

Solutions. Glue is \( f(n) = n^2 \); reference is \( g(n) = n^{\log_2 4} = n^{1/2} \). We have
\[
\frac{f(n)}{g(n)} = \frac{n^2}{n^{0.5+\epsilon}} \to \infty,
\]
for any \( 0 < \epsilon < 1.5 \). Hence \( f = \omega(g_+) \), which implies \( f = \Omega(g_+) \). Case (c) may apply, but we need \( 0 < c < 1 \) with \( af(n/b) \leq cf(n) \) for all large \( n \). Solving,
\[
af(n/b) = 2f(n/4) = 2 \left( \frac{n}{4} \right)^2 = \frac{2n^2}{16} = \frac{1}{8}n^2 = \frac{1}{8}f(n).
\]

\( c = 1/8 \) works. We conclude \( T(n) = \Theta(f) = \Theta(n^2) \).

3. \( T(n) = 4T(n/2) + n \lg n \)

Solutions. Glue is \( f(n) = n \lg n \); reference is \( g(n) = n^{\log_2 4} = n^2 \). We have
\[
\frac{f(n)}{g(n)} = \frac{n \lg n}{n^{2-\epsilon}} \to 0,
\]
for any \( 0 < \epsilon < 1 \). Consequently, \( f = o(g_-) \), which implies \( f = O(g_-) \) and enables case (a) of the master template. We conclude \( T(n) = \Theta(g(n)) = \Theta(n^2) \).

4. \( T(n) = 27T(n/3) + 54n^3 \)

Solutions. Glue is \( f(n) = 54n^3 \); reference is \( n^{\log_3 27} = n^3 \). We have \( f = \Theta(g) \), which enables case (b) of the master template. We conclude \( T(n) = \Theta(n^3 \lg n) \).

5. Prove: For a particular recurrence \( T(n) = aT(n/b) + f(n) \), if one of the master template cases holds, then the other two cases must fail.

Solutions. The glue function is \( f(n) \); the reference function is \( g(n) = n^{\log_b a} \).

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(a) Suppose case (a) holds. Then $f = O(n^{(\log_b a) - \epsilon})$ for some $\epsilon > 0$. Consequently, for some finite $K > 0$, and large $n$,

$$f(n) \leq K n^{(\log_b a) - \epsilon}$$

$$\frac{f(n)}{g(n)} \leq \frac{K n^{(\log_b a) - \epsilon}}{n^{\log_b a}} = \frac{K}{n^\epsilon} \to 0$$

$$f \not\in \Omega(g)$$

$$f \not\in \Theta(g),$$

and case (b) fails to apply. Also, for any exponentially enhanced reference $g_+(n) = n^{(\log_b a) + \delta}$, we have

$$\frac{f(n)}{g_+(n)} \leq \frac{K n^{(\log_b a) - \epsilon}}{n^{(\log_b a) + \delta}} = \frac{K}{n^{\delta + \epsilon}} \to 0$$

$$f \not\in \Omega(g_+),$$

for any $\delta > 0$ associated with the enhanced reference. Therefore case (c) fails to apply.

(b) Suppose case (b) holds. Then $f = \Theta(g) = \Theta(n^{(\log_b a)})$, and there exist constants $0 < K_1 \leq K_2 < \infty$ such that

$$K_1 g(n) \leq f(n) \leq K_2 g(n)$$

for all sufficiently large $n$. For any exponentially diminished reference $g_-(n) = n^{(\log_b a) - \epsilon}$ and large $n$, we have

$$\frac{f(n)}{g_-(n)} \geq \frac{K_1 g(n)}{g_-(n)} = \frac{K_1 n^{\log_b a}}{n^{(\log_b a) - \epsilon}} = K_1 n^\epsilon \to \infty$$

$$f \not\in O(n^{(\log_b a) - \epsilon}),$$

for any $\epsilon > 0$. Therefore case (a) does not apply. Also, for any exponentially enhanced reference $g_+(n) = n^{(\log_b a) + \epsilon}$ and large $n$, we have

$$\frac{f(n)}{g_+(n)} \leq \frac{K_2 n^{\log_b a}}{n^{(\log_b a) + \epsilon}} = \frac{K_2}{n^\epsilon} \to 0$$

$$f \not\in \Omega(n^{(\log_b a) + \epsilon}),$$

for any $\epsilon > 0$. Therefore case (c) does not apply.

(c) Finally, suppose case (c) holds. Then $f = \Omega(n^{(\log_b a) + \epsilon})$ for some $\epsilon > 0$. Consequently, for some $K > 0$ and large $n$,

$$f(n) \geq K n^{(\log_b a) + \epsilon}$$

$$\frac{f(n)}{g(n)} \geq \frac{K n^{(\log_b a) + \epsilon}}{n^{\log_b a}} = K n^\epsilon \to \infty$$

$$f \not\in O(g)$$

$$f \not\in \Theta(g),$$

and therefore case (b) does not apply. Also, for any exponentially diminished reference $g_-(n) = n^{(\log_b a) - \delta}$ and large $n$, we have

$$\frac{f(n)}{g_-(n)} \geq \frac{K n^{(\log_b a) + \epsilon}}{n^{(\log_b a) - \delta}} = K n^{\epsilon + \delta} \to \infty$$

$$f \not\in O(g_-),$$

for any exponentially diminished $g_-$, regardless of the $\delta > 0$. Therefore case (a) fails to apply. ■