1. Let
\[ f(n) = \sum_{k=1}^{n} \frac{6}{k^2}. \]

To show that \( f(n) = \Theta(1) \), we need numbers \( N, K_1, K_2 \) such that \( n \geq N \Rightarrow 0 < K_1 \leq f(n) \leq K_2 < \infty \). Find values for \( N, K_1, K_2 \) and show how these numbers confirm \( f(n) = \Theta(1) \).

2. For \( 0 < k < n \), prove that
\[ \binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1}. \]

3. Alien Alice claims to have extrasensory perception. Baysian Bob places a prior probability \( 10^{-6} \) on the truth of Alice’s assertion. Bob isolates Alice in a room having no possibility of communication with the outside world and then, in another distant room, flips a fair coin \( n \) times. Alice correctly reports the correct sequence of heads and tails. Following Bayes’ Theorem, Bob evaluates a posterior probability that Alice has extrasensory perception. Clearly, a large \( n \) forces Bob to accept a larger probability that Alice is indeed clairvoyant. What is the smallest value of \( n \) such that Bob would calculate a posterior probability greater than or equal to \( 1/2 \)?

Most of the credit for this problem will be given for correctly setting up the equation for \( n \); the balance will be for finding an expression for \( n \) that needs only arithmetic to finish the calculation. Use the reverse side of this page for your calculations.
Solutions.

1. Let

\[ f(n) = \sum_{k=1}^{n} \frac{6}{k^2}. \]

To show that \( f(n) = \Theta(1) \), we need numbers \( N, K_1, K_2 \) such that \( n \geq N \Rightarrow 0 < K_1 \leq f(n) \leq K_2 < \infty \). Find values for \( N, K_1, K_2 \) and show how these numbers confirm \( f(n) = \Theta(1) \).

For any \( n \geq 1 \), the first term of the sequence is six and the remaining terms are positive fractions. Therefore for any \( n \geq 1 \), we have

\[
6 \leq \sum_{k=1}^{n} \frac{6}{k^2} \leq 6 + \sum_{k=2}^{n} \frac{6}{k(k-1)} = 6 + \sum_{k=2}^{n} \left( \frac{6}{k-1} - \frac{6}{k} \right) = 6 + 6 \left( \sum_{k=2}^{n} \frac{1}{k-1} - \sum_{k=2}^{n} \frac{1}{k} \right) 
\]

\[
= 6 + 6 \left( \sum_{k=1}^{n-1} \frac{1}{k} - \sum_{k=2}^{n} \frac{1}{k} \right) = 6 + 6 \left( 1 + \sum_{k=2}^{n-1} \frac{1}{k} - \sum_{k=2}^{n-1} \frac{1}{k} - \frac{1}{n} \right) = 6 + 6 \left( 1 - \frac{1}{n} \right) \leq 12.
\]

We let \( N = 1, K_1 = 1, K_2 = 12 \), and the above calculations confirm that \( n \geq N \Rightarrow 0 < K_1 \leq f(n) \leq K_2 < \infty \), which in turn implies \( f(n) = \Theta(1) \).

2. For \( 0 < k < n \), prove that

\[
\left( \begin{array}{c} n \\ k \end{array} \right) = \frac{n}{k} \frac{(n-1)!}{(k-1)!}.
\]

By direct calculation,

\[
\left( \begin{array}{c} n \\ k \end{array} \right) = \frac{n!}{k!(n-k)!} = \frac{n \cdot (n-1)!}{[k \cdot (k-1)]!(n-k)!} = \frac{n}{k} \frac{(n-1)!}{(k-1)![(n-1)-(k-1)]!} = \frac{n}{k} \left( \begin{array}{c} n-1 \\ k \end{array} \right).
\]

3. Alien Alice claims to have extrasensory perception. Baysian Bob places a prior probability \( 10^{-6} \) on the truth of Alice’s assertion. Bob isolates Alice in a room having no possibility of communication with the outside world and then, in another distant room, flips a fair coin \( n \) times. Alice correctly reports the correct sequence of heads and tails. Following Bayes’ Theorem, Bob evaluates a posterior probability that Alice has extrasensory perception. Clearly, a large \( n \) forces Bob to accept a larger probability that Alice is indeed clairvoyant. What is the smallest value of \( n \) such that Bob would calculate a posterior probability greater than or equal to 1/2? Let \( E \) be the event that Alice has extrasensory perception; let \( R \) be the event that she correctly reports the coin flip sequence. With an overscore indicating the complement of an event, we have

\[
P(E) = 1.0 \times 10^{-6}
\]

\[
P(\overline{E}) = 1 - P(E) = 0.999999
\]

\[
P(R|E) = 1.0
\]

\[
P(R|\overline{E}) = \left( \frac{1}{2} \right)^n
\]

\[
P(E|R) = \frac{P(R|E)P(E)}{P(R|E)P(E) + P(R|\overline{E})P(\overline{E})} = \frac{10^{-6}}{10^{-6} + 0.999999(1/2)^n} = \frac{1}{1 + 999999/2^n}.
\]

To have the posterior \( P(E|R) \geq 1/2 \), we need

\[
\frac{999999}{2^n} \leq 1
\]

\[
2^n \geq 999999.
\]

Since \( 2^{20} = (1024)^2 = 1048576 > 999999 \), and \( 2^{19} \) is half that value, which is less than 999999, we conclude that integer \( n \geq 20 \).