Math 304: Assignment 1

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1

Let m and n be positive integers. Let A be an arbitrary $n \times m$ matrix. There are 12 vector spaces related to the matrix A that are of interest. Those are

$$\begin{array}{lll}
\mathrm{Col}\, A \subseteq \mathbb{R}^n & \mathrm{Row}\, A \subseteq \mathbb{R}^m & \mathrm{Nul}\, A \subseteq \mathbb{R}^m \\
(\mathrm{Col}\, A)^\perp \subseteq \mathbb{R}^n & (\mathrm{Row}\, A)^\perp \subseteq \mathbb{R}^m & (\mathrm{Nul}\, A)^\perp \subseteq \mathbb{R}^m \\
\mathrm{Col}(A^T) \subseteq \mathbb{R}^m & \mathrm{Row}(A^T) \subseteq \mathbb{R}^n & \mathrm{Nul}(A^T) \subseteq \mathbb{R}^n \\
\big(\mathrm{Col}(A^T)\big)^\perp \subseteq \mathbb{R}^m & \big(\mathrm{Row}(A^T)\big)^\perp \subseteq \mathbb{R}^n & \big(\mathrm{Nul}(A^T)\big)^\perp \subseteq \mathbb{R}^n
\end{array}$$

Let the matrices A and A^T contain entries as follows:

$$A = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix} A^T = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

If no distinction is observed between the vectors and column vectors, two useful identities between the listed vector spaces can be established from the definition of the transpose of a matrix.

$$\mathrm{Col}\, A = \mathrm{Row}(A^T)$$
$$\mathrm{Col}(A^T) = \mathrm{Row}\, A$$

Substituting these identities into our original list and eliminating copies gives the new list:

$$\begin{array}{ll}
\mathrm{Col}\, A = \mathrm{Row}\, A^T & \mathrm{Nul}\, A \\
(\mathrm{Col}\, A)^\perp = (\mathrm{Row}\, A^T)^\perp & (\mathrm{Nul}\, A)^\perp \\
\mathrm{Col}(A^T) = \mathrm{Row}(A) & \mathrm{Nul}(A^T) \\
\big(\mathrm{Col}(A^T)\big)^\perp = \big(\mathrm{Row}(A)\big)^\perp & \big(\mathrm{Nul}(A^T)\big)^\perp
\end{array}$$

Let the column vectors of A be $\{\vec{a_1}, \vec{a_2}, ..., \vec{a_m}\}$. Then a vector $\vec{x}$ is in $(\mathrm{Col}\, A)^\perp$ if and only if $\vec{x} \cdot \vec{a_j} = 0$ for $1 \leq j \leq m$. Rewriting this as a system of equations we have:

$$a_{11}x_1 + a_{12}x_2 ... + a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 ... + a_{1n}x_n = 0$$

$$.$$

$$.$$

$$.$$

$$a_{n1}x_1 + a_{n2}x_2 ... + a_{mn}x_n = 0$$

Rewriting the system as a matrix equation we have:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \vec{0} \Leftrightarrow A^T\vec{x} = \vec{0}$$

So, any vector $\vec{x}$ is in $(\mathrm{Col}\, A)^\perp$ if and only if $\vec{x} \in \mathrm{Nul}(A^T)$. Since no special assumptions were made about A we can conclude that:

$$(\mathrm{Col}\, A)^\perp = \mathrm{Nul}(A^T)$$
$$(\mathrm{Col}\, A^T)^\perp = \mathrm{Nul}(A)$$

Also notice by these identities that:

$$(\mathrm{Nul}\, A^T)^\perp = ((\mathrm{Col}\, A)^\perp)^\perp = \mathrm{Col}\, A$$

$$(\mathrm{Nul}\, A)^\perp = ((\mathrm{Col}\, A^T)^\perp)^\perp = \mathrm{Col}\, A^T$$

Substituting these identities into our shortened list and eliminating identical entries we have:

$$\mathrm{Col}\, A = \mathrm{Row}\, A^T = (\mathrm{Nul}\, A^T)^\perp$$
$$\mathrm{Nul}\, A^T = (\mathrm{Col}\, A)^\perp = (\mathrm{Row}\, A^T)^\perp$$
$$\mathrm{Col}(A^T) = \mathrm{Row}(A) = (\mathrm{Nul}\, A)^\perp$$
$$\mathrm{Nul}\, A = \big(\mathrm{Col}(A^T)\big)^\perp = \big(\mathrm{Row}(A)\big)^\perp$$

By inspection one finds that for the matrix $Q = \left(\begin{smallmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{smallmatrix}\right)$, the four vector spaces on our list are in fact distinct. So we conclude that for some matrix A there can be as many as four distinct subspaces of those on the original list and there are at most four distinct subspaces contained on the original list.

When $m = n$ it is possible that $A = A^T$ as in the case of I_n. In these cases $\operatorname{Col} A = \operatorname{Col}(A^T)$, and $\operatorname{Nul} A^T = \operatorname{Nul} A$, reducing the number of distinct vector spaces from the list to 2. So we conclude that there can be as few as 2 distinct vector spaces contained on the original list.

We now prove by contradiction that for a vector space W in $\mathbb{R}^n$, $W \neq W^\perp$.

Proof. Suppose there exists a vector space W in $\mathbb{R}^n$ such that $W = W^\perp$. Then every vector in W is orthogonal to itself. By the definition of orthogonality and a property of the dot product ($\vec{u} \cdot \vec{u} = \vec{0}$ if and only if $\vec{u} = 0$), the only vector orthogonal to itself is $\vec{0}$. This implies $W = \vec{0}$. Since every vector in R^n is orthogonal to $\vec{0}$, we have $W^\perp = \mathbb{R}^n$. By our assumption this gives $\mathbb{R}^n = \vec{0}$ which is clearly not true. So we conclude by way of contradiction that for any vector space in $\mathbb{R}^n$, $W \neq W^\perp$. □

Since the original list contains the orthogonal complement of each vector space on the list, we conclude that there are at least two distinct vector spaces on the list.

2

Let $n \in \mathbb{N}$. In this problem below we consider the reverse identity matrix, J_n.

$$J_1 = \begin{bmatrix} 1 \end{bmatrix}, \; J_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; J_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \; \cdots, \; J_n = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \iddots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{bmatrix}$$

Let 0_n be an n x n matrix with all zeros for entries. Then for even n, J_n can be represented in the form of a block matrix as follows: $J_n = \left(\begin{smallmatrix} 0_{\frac{n}{2}} & J_{\frac{n}{2}} \\ J_{\frac{n}{2}} & 0_{\frac{n}{2}} \end{smallmatrix}\right)$. Let $P_n = \left(\begin{smallmatrix} I_{\frac{n}{2}} & J_{\frac{n}{2}} \\ J_{\frac{n}{2}} & (-1)I_{\frac{n}{2}} \end{smallmatrix}\right)$ and $D_n = \left(\begin{smallmatrix} I_{\frac{n}{2}} & 0_{\frac{n}{2}} \\ 0_{\frac{n}{2}} & (-1)I_{\frac{n}{2}} \end{smallmatrix}\right)$. Notice that $J_n J_n = I_n$. So we have the following results:

$$(1) \quad J_n P_n = \begin{bmatrix} 0_{\frac{n}{2}} & J_{\frac{n}{2}} \\ J_{\frac{n}{2}} & 0_{\frac{n}{2}} \end{bmatrix} \begin{bmatrix} I_{\frac{n}{2}} & J_{\frac{n}{2}} \\ J_{\frac{n}{2}} & (-1)I_{\frac{n}{2}} \end{bmatrix} = \begin{bmatrix} J_{\frac{n}{2}} J_{\frac{n}{2}} & J_{\frac{n}{2}}(-1)I_{\frac{n}{2}} \\ J_{\frac{n}{2}} I_{\frac{n}{2}} & J_{\frac{n}{2}} J_{\frac{n}{2}} \end{bmatrix} = \begin{bmatrix} I_{\frac{n}{2}} & (-1)J_{\frac{n}{2}} \\ J_{\frac{n}{2}} & I_{\frac{n}{2}} \end{bmatrix}$$

$$(2) \quad P_n D_n = \begin{bmatrix} I_{\frac{n}{2}} & J_{\frac{n}{2}} \\ J_{\frac{n}{2}} & (-1)I_{\frac{n}{2}} \end{bmatrix} \begin{bmatrix} I_{\frac{n}{2}} & 0_{\frac{n}{2}} \\ 0_{\frac{n}{2}} & (-1)I_{\frac{n}{2}} \end{bmatrix} = \begin{bmatrix} I_{\frac{n}{2}} I_{\frac{n}{2}} & J_{\frac{n}{2}}(-1)I_{\frac{n}{2}} \\ I_{\frac{n}{2}} J_{\frac{n}{2}} & (-1)I_{\frac{n}{2}}(-1)I_{\frac{n}{2}} \end{bmatrix} = \begin{bmatrix} I_{\frac{n}{2}} & (-1)J_{\frac{n}{2}} \\ J_{\frac{n}{2}} & I_{\frac{n}{2}} \end{bmatrix}$$

By (1) and (2) we have $J_n P_n = P_n D_n$. Since D_n is diagonal we now show that $P_n(\frac{1}{2}P_n) = I_n$, to prove that for even n, J_n is diagonalizable.

$$\frac{1}{2}P_nP_n = \frac{1}{2}\begin{bmatrix} I_{\frac{n}{2}} & J_{\frac{n}{2}} \\ J_{\frac{n}{2}} & (-1)I_{\frac{n}{2}} \end{bmatrix}\begin{bmatrix} I_{\frac{n}{2}} & J_{\frac{n}{2}} \\ J_{\frac{n}{2}} & (-1)I_{\frac{n}{2}} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} (I^2_{\frac{n}{2}} + J^2_{\frac{n}{2}}) & (I_{\frac{n}{2}}J_{\frac{n}{2}} - I_{\frac{n}{2}}J_{\frac{n}{2}}) \\ (J^2_{\frac{n}{2}} - I^2_{\frac{n}{2}}) & (J^2_{\frac{n}{2}} + I^2_{\frac{n}{2}}) \end{bmatrix} = I_n$$

For a non-square $n \times m$ matrix containing all zeros as entries we introduce the notation $0_{n\times m}$. For square matrices containing all zeros as entries we maintain our earlier convention. This allows us to represent J_n where n is odd and $n > 1$, as the block matrix displayed below. Furthermore, let the matrices Q_n, V_n be as follows:

$$J_n = \begin{bmatrix} 0_{\frac{n-1}{2}} & 0_{\frac{n-1}{2}\times 1} & J_{\frac{n-1}{2}} \\ 0_{1\times\frac{n-1}{2}} & 1 & 0_{1\times\frac{n-1}{2}} \\ J_{\frac{n-1}{2}} & 0_{\frac{n-1}{2}\times 1} & 0_{\frac{n-1}{2}} \end{bmatrix}, Q_n = \begin{bmatrix} I_{\frac{n-1}{2}} & 0_{\frac{n-1}{2}\times 1} & J_{\frac{n-1}{2}} \\ 0_{1\times\frac{n-1}{2}} & 1 & 0_{1\times\frac{n-1}{2}} \\ J_{\frac{n-1}{2}} & 0_{\frac{n-1}{2}\times 1} & (-1)I_{\frac{n-1}{2}} \end{bmatrix} V_n = \begin{bmatrix} I_{\frac{n-1}{2}} & 0_{\frac{n-1}{2}\times 1} & 0_{\frac{n-1}{2}} \\ 0_{1\times\frac{n-1}{2}} & 1 & 0_{1\times\frac{n-1}{2}} \\ 0_{\frac{n-1}{2}} & 0_{\frac{n-1}{2}\times 1} & (-1)I_{\frac{n-1}{2}} \end{bmatrix}$$

Since J_n, Q_n, and V_n are conformable for block multiplication, for the sake of space we shall ignore subscripts defining block size in demonstration of the matrix products we are interested in. We have:

$$(1) \quad J_nQ_n = \begin{bmatrix} 0 & 0 & J \\ 0 & 1 & 0 \\ J & 0 & 0 \end{bmatrix}\begin{bmatrix} I & 0 & J \\ 0 & 1 & 0 \\ J & 0 & -I \end{bmatrix} = \begin{bmatrix} J^2 & 0 & -IJ \\ 0 & 1 & 0 \\ JI & 0 & J^2 \end{bmatrix} = \begin{bmatrix} I & 0 & -J \\ 0 & 1 & 0 \\ J & 0 & I \end{bmatrix}$$

$$(2) \quad Q_nV_n = \begin{bmatrix} I & 0 & J \\ 0 & 1 & 0 \\ J & 0 & -I \end{bmatrix}\begin{bmatrix} I & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -I \end{bmatrix} = \begin{bmatrix} I & 0 & J(-I) \\ 0 & 1 & 0 \\ J & 0 & -I(-I) \end{bmatrix} = \begin{bmatrix} I & 0 & -J \\ 0 & 1 & 0 \\ J & 0 & I \end{bmatrix}$$

By (1) and (2) we have $J_nQ_n = Q_nV_n$. Since V_n is a diagonal matrix we now demonstrate that Q_n is invertible to complete the proof that J_n is diagonalizable for odd n ($n = 1$ is trivial since J_1 is a diagonal matrix). Let

$$Z_n = \frac{1}{2}\begin{bmatrix} I_{\frac{n-1}{2}} & 0_{\frac{n-1}{2}\times 1} & J_{\frac{n-1}{2}} \\ 0_{1\times\frac{n-1}{2}} & 2 & 0_{1\times\frac{n-1}{2}} \\ J_{\frac{n-1}{2}} & 0_{\frac{n-1}{2}\times 1} & (-1)I_{\frac{n-1}{2}} \end{bmatrix}$$

Then:

$$Q_nZ_n = \begin{bmatrix} I & 0 & J \\ 0 & 1 & 0 \\ J & 0 & -I \end{bmatrix}\frac{1}{2}\begin{bmatrix} I & 0 & J \\ 0 & 2 & 0 \\ J & 0 & -I \end{bmatrix} = \frac{1}{2}\begin{bmatrix} I^2 + J^2 & 0 & IJ + J(-I) \\ 0 & 2 & 0 \\ JI + (-IJ) & 0 & J^2 + (-I)(-I) \end{bmatrix} = I_n$$

Since J_n is diagonalizable for all even n and all odd n, J_n is diagonalizable for all n.

3

In this problem we consider three kinds of $n \times n$ matrices:

$$L_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}, \quad U_n = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \quad M_n = \begin{bmatrix} n & n-1 & \cdots & 2 & 1 \\ n-1 & n-1 & \cdots & 2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 2 & \cdots & 2 & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{bmatrix}.$$

1. There is a simple relationship among matrices L_n, U_n and M_n. By matrix multiplication we calculate (1) $U_n L_n = Mn$.

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} n & n-1 & \cdots & 2 & 1 \\ n-1 & n-1 & \cdots & 2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 2 & \cdots & 2 & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{bmatrix}$$

2. Since L_n and U_n are triangular matrices it is easy to calculate their determinants.

$$det(L_n) = 1 \cdot 1 \cdot ..... \cdot 1 = 1$$
$$det(U_n) = 1 \cdot 1 \cdot ..... \cdot 1 = 1$$

By (1) we have $det(M_n) = det(U_n L_n) = det(U_n)det(L_n) = 1$.

3.

$$U_n^{-1} = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & -1 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \quad L_n^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix}$$

$$M_n^{-1} = (U_n L_n)^{-1} = L_n^{-1} U_n^{-1} =$$

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & -1 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & -1 \\ 0 & 0 & \cdots & -1 & 2 \end{bmatrix}$$

4

Prove that $\text{Col}(A^T) = \text{Col}(A^T A)$.

Proof. First we show that $\text{Nul}(A) = \text{Nul}A^T A$. For $\vec{x} \in \text{Nul}(A)$, $A\vec{x} = \vec{0}$. Right multiplying both sides by A^T gives $A^T A\vec{x} = \vec{0}$. So, (1) if $\vec{x} \in \text{Nul}(A)$ then $\vec{x} \in \text{Nul}(A^T A)$.

For $\vec{x} \in \text{Nul}(A^T A)$, $A^T A\vec{x} = \vec{0}$. We have:

$$A^T A\vec{x} = \vec{0} \Leftrightarrow \vec{x}^T A^T A\vec{x} = \vec{0} \Leftrightarrow (A\vec{x})^T A\vec{x} = \vec{0} \Leftrightarrow A\vec{x} \cdot A\vec{x} = \vec{0} \Leftrightarrow \| A\vec{x} \|^2 = \vec{0} \Leftrightarrow \| A\vec{x} \| = \vec{0} \Leftrightarrow A\vec{x} = \vec{0}$$

So, (2) if $\vec{x} \in \text{Nul}(A^T A)$ then $\vec{x} \in \text{Nul}(A)$. By (1) and (2) we have (3) $\text{Nul}(A^T A) = \text{Nul}(A)$. From (3) and the identities established in section 1 we have the following result:

$$\text{Col}(A^T) = (\text{Nul}(A))^\perp = (\text{Nul}(A^T A))^\perp = \text{Col}(A^T A)^T = \text{Col}(A^T A).$$

□

By the above result, the fact $A^T A = (A^T A)^T$, and our previously established identities we now have these useful identities:

$$\text{Col}(A^T) = \text{Row}(A) = (\text{Nul}\, A)^\perp = \text{Col}(A^T A) = \text{Row}(A^T A).$$